earlier studies,[2-4] which, in our opinion, are free of such misconceptions and place the time element discretization of Hamilton's Law on a sound theoretical foundation. This is the aim of the following discussion.

Hamilton's Law,[2,5] in the absence of nonconservative forces, states that

$$\delta \int_{t_0}^{t_f} (T-V)\,\mathrm{d}t - \sum_i \frac{\partial T}{\partial \dot{q}_i}\delta q_i \Big|_{t_0}^{t_f} = 0 \tag{1}$$

and serves as the basis for the present discussion. In Eq. (1), T and V are the kinetic and the potential energy of the system, respectively, and $q_i\,(i=1,\ldots,n)$ are the generalized coordinates. In accordance with the finite element technique, the integral expression of Eq. (1) can be discretized and expressed as a sum over time finite elements.[3,4] By prescribing the same interpolation procedure as in Refs. 3 and 1, the values of q_i, $\dot{q}_i$ may be derived at any instant t at the element in terms of their values at the endpoints of this particular element. For the case of a single particle ($i=1$) having one degree-of-freedom, $u(t)$, this interpolation procedure and summation of the contributions of all of the elements yield the following expression for the integral and the summation term in Eq. (1).

$$H = U^T[(K-B)U-F] \tag{2}$$

where K and B are $(2n+2)\times(2n+2)$ matrices obtained from $(T-V)$ and the second term, respectively, and $U^T = [u_0,\dot{u}_0,u_1,\dot{u}_1,\ldots,u_n,\dot{u}_n]$.

At this stage it is important to repeat the fundamental axiomatic assumption given in Refs. 2 and 3: The physical solution of a given dynamic problem exists and is unique. The word "physical" means here that, although for some cases there may be more than one mathematical solution, for a real dynamic system, only one of them can be realized physically. This solution clearly depends on the system initial values. Once we know the existing solution, the state variables are known at any time, including the initial and the final state variables. In other words, there is a one to one relationship between the initial and the final values of the system. The essence of the above discussion is not changed if, in the sequel, one of these values is assumed to be known rather than being known. Based on the above assumption the following technique can be developed.[2-4] We assume that any possible combination of the initial and final coordinates and velocities are known; hence, their variations are equal to zero. Their number is taken to be equal to the number of the original initial conditions to make the formulation well defined. The reader will notice that q_i and $\dot{q}_i$, although assumed known, are not constrained at t_θ and t_f. Only some of their variations are imposed to be zero. Since these particular variations are no longer arbitrary, the equations they multiply equal some undefined constants, which can be different from zero. After introducing the correct initial values one can calculate these constants or eliminate them. For a one-degree-of-freedom dynamic system having two initial and two final state variables $(u,\dot{u})$, there exist six such possible combinations: $(\delta u_0\ \delta\dot{u}_0;\ \delta u_0\ \delta u_f;\ \delta\dot{u}_0\ \delta\dot{u}_f;\delta u_f\ \delta\dot{u}_f;\ \delta u_0\ \delta\dot{u}_f;\ \delta\dot{u}_0\ \delta u_f)$. A full discussion and exposition of all the possibilities require a full-length paper. The reader interested in details is referred to Refs. 2 and 4. Here we proceed with the possibility that $\delta u_0 = \delta u_f = 0^3$, which is the appropriate case for the paper in comment.

The first variation of Eq. (2) when $\delta u_0 = \delta u_f = 0$ yields the following system of equations,

$$(K-B)U-F=\psi \tag{3}$$

where

$$\psi^T = [\psi_1,0,0,\ldots,0,\psi_2,0]$$

where ψ_1 and ψ_2 are the two undefined constants. Equation (3) cannot be solved unless the initial values, e.g., u_0, $\dot{u}_0$, of the specific problem are introduced. In fact, they can be imposed on Eq. (3) in the same consistent manner as one imposes boundary conditions in standard finite element procedures.[3] It follows that the set (3) presents $2n+2$ equations for the $2n+2$ unknowns,

$$u_1,\dot{u}_1,u_2,\dot{u}_2,\ldots,u_n,\dot{u}_n,\psi_1,\psi_2$$

The absence of ψ_2 in Ref. 1 is the reason for the "unknown" instability appearing there. Clearly, one can eliminate the two equations which correspond to ψ_1 and ψ_2. It follows that the restricted set (3) presents then $2n$ equations for the $2n$ unknowns. This elimination, for this and only this particular possibility $(\delta u_0 = \delta u_f = 0)$ under consideration, is equivalent to imposing

$$\frac{\partial T}{\partial \dot{u}}\delta u \Big|_{t_0}^{t_f}$$

to be zero, i.e., applying Hamilton's Principle, which requires the suppression of the two equations which correspond to u_0 and u_f from the system.

Thus, the same consistent and well defined set of equations for the initial value problem is achieved, either by employing Hamilton's Law or Hamilton's Principle, ruling out the need for mathematical manipulations, contrary to the formulation in Ref. 1. Following these procedures,[2-4] convergent solutions were obtained for the problem of the free oscillator considered by Simkins,[1] as well as for more complicated problems.

References

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Reply by Author to Riff, Weller, and Baruch

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THE lead paragraph of the Comment is seriously lacking in specificity, i.e., if the authors think I have "misconceptions," let them state what they are so that I may have a fair chance at rebuttal. Further, it should not be overlooked that my "attempt" to formulate a time-finite element formulation based on Hamilton's Law of Varying Action was a *successful* attempt in every way. The "mathematical manipulations" to which the authors have

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evidently given little study (or they would not refer to them as "manipulations") are fully justified because of the quasi-singular state of the coefficient matrix. Indeed, for an n degree-of-freedom system, this singular state allows an *arbitrary* deletion of $2n$ rows and columns from this matrix. This is the same result that would be achieved by setting any two variational quantities to zero, e.g., by "assuming that any possible combination of the initial and final coordinates and their velocities are known . . ." However, in reality the known quantities are not arbitrarily chosen and to "assume" or "pretend"[1] that they are is unnecessary. Again, it is the $2n$ singular state of the coefficient matrix that allows arbitrary removal of rows and columns; there is no need to pretend that certain unknown quantities are known. The "mathematical manipulations" to which the authors refer are simply an explicit demonstration of this fact.

There is one aspect of the Comment which is of interest, however; that is, the idea of approximating the displacement and velocity variations using functions different from those used to approximate the displacements and velocities themselves. This permits direct use of Hamilton's Principle to formulate initial value problems, because one can use a family of functions which vanish identically at the end points to approximate the variations without implying any knowledge about the value of the displacements or velocities at the upper end point. I personally have shied away from this practice, preferring to following the usual finite element method of structual mechanics, i.e., the Ritz procedure using the same piecewise basis functions for both the unknown function and its variation. This preference stems from the desire to obtain the exact solution in certain cases. For example, one may actually be presented with an n degree-of-freedom system or equivalently, have knowledge that all of the motion is contained in certain n modes. We know tht in such cases the use of these n-mode shapes as basis functions for both the displacement and its variation will give the exact solution. If the variations are expressed in terms of functions from a different family the exact result is not assured. However, one sees increased use of the later procedure in other fields (e.g., fluid mechanics) and as long as convergence is achieved, this procedure may have a future.

As a final note, if the authors of the above comment persist in the concept that there are 6^n correct formulations of Hamilton's Principle or Law,[1] then they should realize that there are really many more formulations, e.g., in a finite element representation one can "pretend" that *any* two variations vanish—not just combinations formed from the end point variations.

References

[1]Baruch, M. and Riff, R., "Hamilton's Principle, Hamilton's Law – 6^n Correct Formulations," *AIAA Journal,* Vol. 20, May 1982, pp. 687-692.